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LETTER TO THE EDITOR

New exact solutions of the complex Lorenz equations

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Abstract. We present two classes of particular exact solutions to the complex Lorenz equations. These solutions possess the Painlevé property. The parameter range for which the above solutions hold includes as special cases two of the three parameter ranges for which the general solution of the complex Lorenz equations has the Painlevé property.

The complex Lorenz equations (CLE) proposed by Fowler, Gibbon and McGuinness [1-3] are

$$\dot{x} = \sigma(y - x) \tag{1a}$$

$$\dot{y} = rx - xz - ay \tag{1b}$$

$$\dot{z} = -bz + \tfrac{1}{2}(x^*y + xy^*) \tag{1c}$$

the dot denoting differentiation with respect to time t. The parameters b, σ, r, a are defined by

$$b > 0 \qquad \sigma > 0 \qquad r = r_1 + \mathrm{i}r_2 \qquad r_1 > 0$$
$$r_2 > 0 \qquad a = 1 - \mathrm{i}e \qquad e > 0. \tag{2}$$

The real Lorenz equations (RLE) are recovered from ($1a$–c) by setting $r_2 = e = 0$ and considering real $x(t)$, $y(t)$, since $z(t)$ is in any case real.

A Painlevé analysis of the CLE has been carried out [4-6] and it has been found that the general solution of the system ($1a$–c) possesses the Painlevé property (only poles as movable singularities in the complex t plane) for

$$\begin{array}{lllllll}
\text{(a)} & \sigma = \frac{1}{2}, & r_1 = e^2/2 & b = 1 & r_2 = e/2 & e \text{ arbitrary} & (3a) \\
\text{(b)} & \sigma = 1 & r_1 = e^2/4 + \frac{1}{9} & b = 2 & r_2 = 0 & e \text{ arbitrary} & (3b) \\
\text{(c)} & \sigma = \frac{1}{3} & r_1 = \text{arbitrary} & b = 0 & r_2 - e & e \text{ arbitrary.} & (3c)
\end{array}$$

Very recently [6] Roekaerts has obtained by applying the REDUCE program DISSYS developed by Schwarz [7] all constants of the motion of ($1a$–c) of the form

$$F(x_1, x_2, x_3, x_4, z, t) = \exp(c_0 t) P(x_1, x_2, x_3, x_4, z) \tag{4}$$

where

$$x = x_1 + \mathrm{i}x_2 \qquad y = x_3 + \mathrm{i}x_4 \qquad (x_1, x_2, x_3, x_4\text{: real}) \tag{5}$$

c_0 is a constant depending on σ, b, r_1, r_2, e, and P is a polynomial of at most fourth order in its arguments. The results concerning the RLE are regained by letting $e = r_2 = 0$ in ($3a$–c) and (4).

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In the present letter we proceed one step further and construct new exact particular solutions to the CLE. These solutions possess the Painlevé property and are valid for certain ranges of parameters σ, b, r_1, r_2 and $e>0$. The above parameter ranges include (3a–c). By a simple manipulation we obtain from (1a–c)

$$\ddot{x}+(\sigma+a)\dot{x}+\sigma(a-r)x=-\sigma xz \tag{6a}$$

$$\dot{z}+bz=|x|^2+\frac{\mathrm{d}(x^*x)}{\mathrm{d}t}\frac{1}{2\sigma} \tag{6b}$$

$$y=\dot{x}/\sigma+x. \tag{6c}$$

The systems (1a–c) and (6a–c) are equivalent. From (6b) we deduce

$$z(t)=C\exp(-bt)+\frac{|x|^2}{2\sigma}+\exp(-bt)\left(1-\frac{b}{2\sigma}\right)\int|x|^2\exp(bt)\,\mathrm{d}t \tag{7}$$

$C=$ constant. Let now $C=0$ and

$$b=2\sigma. \tag{8}$$

Owing to the choice $C=0$ and equation (8), (7) becomes

$$z(t)=|x|^2/2\sigma \tag{9}$$

and by virtue of (9), (6a) yields

$$\ddot{x}+(\sigma+a)\dot{x}+\sigma(a-r)x=-x|x|^2/2. \tag{10}$$

We introduce the transformation

$$x(t)=u(\xi)v(t)\qquad \xi=\xi(t)\qquad u'(\xi)=\mathrm{d}u/\mathrm{d}\xi. \tag{11}$$

Transformation (11) is widely applicable in the theory of differential equations and appears also in the method of symmetry reduction [8] of partial differential equations. Thus (10) becomes with $\dot{\xi}=\mathrm{d}\xi/\mathrm{d}t$:

$$u''\dot{\xi}^2v+u'[\ddot{\xi}v+2\dot{\xi}\dot{v}+(\sigma+a)\dot{\xi}v]+u[\ddot{v}+(\sigma+a)\dot{v}+\sigma(a-r)v]=-u|u|^2|v|^2v/2. \tag{12}$$

We now seek to determine $\xi(t)$ and $v(t)$ such that

$$\ddot{\xi}v+2\dot{\xi}\dot{v}+(\sigma+a)\dot{\xi}v=0 \tag{13a}$$

$$\ddot{v}+(\sigma+a)\dot{v}+\sigma(a-r)v=0 \tag{13b}$$

$$(\dot{\xi})^2=|v|^2. \tag{13c}$$

After solving the system (13a) and (13b) we may satisfy (13c). A rather lengthy calculation yields that if the relations

$$r_1=1-\frac{8(\sigma+1)^2-9e^2}{36\sigma}\qquad r_2=\frac{e(1-\sigma)}{2\sigma} \tag{14a}$$

$$\sigma<1\qquad 9e^2+20\sigma-8\sigma^2-8>0 \tag{14b}$$

are valid, condition (14b) deriving from $r_1>0$, $r_2>0$, then

$$\xi(t)=-\frac{3}{\sigma+1}\exp[-(\sigma+1)t/3]\qquad v(t)=\exp\left[t\left(\frac{\mathrm{i}e}{2}-\frac{\sigma+1}{3}\right)\right] \tag{15}$$

and (10) is transformed into

$$u''(\xi) = -u(\xi)|u(\xi)|^2/2. \tag{16}$$

We distinguish two cases.

Case (a). $u(\xi) = real$. Equation (16) is immediately integrable in terms of the Jacobian elliptic function $\mathrm{cn}(\xi, \sqrt{2})$. In fact we get

$$u(\xi) = (4G)^{1/4}\,\mathrm{cn}[G^{1/4}(G_1 - \xi), \sqrt{2}] \tag{17}$$

G, G_1 being positive constants. By virtue of (11) we may write

$$x(t) = (4G)^{1/4}\exp\left[t\left(\frac{\mathrm{i}e}{2} - \frac{\sigma+1}{3}\right)\right]\mathrm{cn}\left[G^{1/4}\left(G_1 + \frac{3}{\sigma+1}\exp[-(\sigma+1)t/3]\right), \sqrt{2}\right] \tag{18}$$

the functions $y(t)$, $z(t)$ being easily written down by means of (6c) and (9) respectively. We now make the following observations.

(I) The set of functions (x, y, z) determined through (6c), (9) and (18) comprises a particular exact solution of the system (1a–c) since it has only two constants of integration, G and G_1, instead of the necessary five, cf (5). The above solution holds provided (8), (14a) and (14b) are fulfilled.

(II) For $e = 0$ we regain from (14a) and (14b) and (18) the results for the RLE [9].

(III) To the best of our knowledge the only other particular exact solution of the CLE known up to now is the periodic solution [2]

$$x(t) = A\exp(\mathrm{i}ft) \qquad y(t) = \left(1 + \frac{\mathrm{i}f}{\sigma}\right)A\exp(\mathrm{i}ft) \qquad z(t) = |A|^2/b$$
$$f = \sigma(e + r_2)/(\sigma+1). \tag{19}$$

The amplitude A of the above exact limit cycle solution equals $b(r_1 - r_{1c})$, where r_{1c} is given by (30) below. Thus solution (19) is valid for $r_1 > r_{1c}$, whereas our new exact solution holds for $r_1 < r_{1c}$, as (14a) shows. We conclude that the limit cycle solution (19) and the new damped oscillation found here *never* coexist.

(IV) The elliptic function appearing in (18) has for complex t only simple poles as movable singularities. Therefore the Painlevé property of the solution (x, y, z) in (I) above is established. A glance at (14a) and (14b) and $b = 2\sigma$ reveals that they include (3a) and (3b) as special cases.

Case (b). $u(\xi) = complex$. Let

$$u(\xi) = u_1(\xi) + \mathrm{i}u_2(\xi). \tag{20}$$

Owing to (20), (16) becomes

$$u_1'' = -u_1(u_1^2 + u_2^2)/2 \tag{21a}$$

$$u_2'' = -u_2(u_1^2 + u_2^2)/2. \tag{21b}$$

The trivial solutions of the system (21a) and (21b), (a) $u_1 = 0$ and (b) $u_1 = u_2$, lead immediately, apart from an inessential phase factor, to precisely the same analytic form for $x(t)$ as in (18). To find non-trivial analytic solutions of (21a) and (21b) we represent u_1, u_2 as

$$u_1(\xi) = r(\xi)\cos\varphi(\xi) \qquad u_2(\xi) = r(\xi)\sin\varphi(\xi). \tag{22}$$

On inserting (22) into (21a) and (21b) we obtain

$$\cos\varphi(\xi)[r'' - (\varphi')^2 r + r^3/2] - \sin\varphi(\xi)[2r'\varphi' + r\varphi''] = 0$$
$$\sin\varphi(\xi)[r'' - (\varphi')^2 r + r^3/2] + \cos\varphi(\xi)[2r'\varphi' + r\varphi''] = 0. \tag{23}$$

Equations (23) are satisfied if

$$r'' - (\varphi')^2 r + r^3/2 = 0 \tag{24a}$$

$$2r'\varphi' + r\varphi'' = 0. \tag{24b}$$

Equation (24*b*) yields

$$\varphi' = C_3/r^2 \qquad C_3 = \text{constant.} \tag{25}$$

Thus (24*a*) becomes

$$r'' - C_3^2/r^3 + r^3/2 = 0. \tag{26}$$

Integration of (26) gives (27):

$$\int \frac{\mathrm{d}w}{(-w^3 + 8C_1 w - 4C_3^2)^{1/2}} = \xi + C_2 \qquad r^2 = w. \tag{27}$$

The elliptic integral in (27) can be treated in a standard way. Since $r(\xi)$ is real, we must place restrictions on C_1, C_3 so that the polynomial $-w^3 + 8C_1 w - 4C_3^2 > 0$. The respective algebraic investigation is easily carried out. Here we shall give the final result obtained by inverting the elliptic integral in (27)

$$r(\xi) = \left\{ a - (a-b)\,\mathrm{sn}^2\left[(\xi + C_2)\left(\frac{a}{2}\right)^{1/2}, \left(\frac{a-b}{2a}\right)^{1/2} \right] \right\}^{1/2} \tag{28}$$

the constants a, b, with $a - b > 0$, depending on C_1, C_3 and $\mathrm{sn}(\xi)$ being the Jacobian elliptic function. The angle $\varphi(\xi)$ can be determined by a quadrature from (25), but it is of no further importance since it is only a phase angle. By virtue of (11), (15), (20), (22) and (28) then $x(t)$, $y(t)$ and $z(t)$ can be written down, for instance

$$\begin{aligned} x(t) = \exp&\left[t\left(\frac{\mathrm{i}e}{2} - \frac{\sigma+1}{3}\right) + \mathrm{i}\varphi \right] \\ &\times \left\{ a\,\mathrm{cn}^2\left[\left(C_2 - \frac{3}{\sigma+1}\exp\left(-\frac{(\sigma+1)t}{3}\right)\right)\left(\frac{a}{2}\right)^{1/2}, \left(\frac{a-b}{2a}\right)^{1/2} \right] \right. \\ &\left. + b\,\mathrm{sn}^2\left[\left(C_2 - \frac{3}{\sigma+1}\exp\left(-\frac{(\sigma+1)t}{3}\right)\right)\left(\frac{a}{2}\right)^{1/2}, \left(\frac{a-b}{2a}\right)^{1/2} \right] \right\}^{1/2}. \end{aligned} \tag{29}$$

Again, we obtain a damped oscillation which has a different analytic form than that in (18).

We now summarise our results.

(1) We have found two new classes of particular exact solutions of the CLE. Both classes possess the Painlevé property and describe a damped oscillation.

(2) According to Fowler *et al* [2] r_1 is used as the bifurcation parameter and

$$r_{1c} = 1 + \frac{(e + r_2)(e - \sigma r_2)}{(\sigma+1)^2} \tag{30}$$

is its value at the stability limit. In our case (4.1) shows that $r_1 < r_{1c}$, which implies that the origin $x = y = z = 0$ is stable. Clearly solutions (18) and (29) conform with that statement.

(3) According to Yoshida [10] the Hamiltonian

$$H = (p_1^2 + p_2^2)/2 + (q_1^4 + q_2^4)/4 + \varepsilon q_1^2 q_2^2/2 \tag{31}$$

with equations of motion

$$q_1'' = -q_1(q_1^2 + \varepsilon q_2^2) \tag{32a}$$

$$q_2'' = -q_2(q_2^2 + \varepsilon q_1^2) \tag{32b}$$

is integrable only for $\varepsilon = 0, 1, 3$. The system of (32*a*) and (32*b*) goes over into the system (21*a*) and (21*b*) for $\varepsilon = 1$, $q_1 = u_1/\sqrt{2}$, $q_2 = u_2/\sqrt{2}$. A byproduct of the present work, therefore, is the exact analytic solution of (32*a*) and (32*b*) for $\varepsilon = 1$, i.e. for one of the three integrable cases of the Hamiltonian (31).

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